Invariant Relations for the Resistively Induced Plasma Flow Out of Tokamaks Under the Influence of Nonlinear Flow Terms and Ion Viscosity

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The resistively induced plasma motion across the magnetic surface of a tokamak is considered taking into account nonlinear inertial flow terms and/or ion viscosity. It is shown that certain relations involving the plasma flow F out of the tokamak remain invariant if one goes from flows without these terms to flows including them. It is concluded that they can affect the flow F only indirectly by rearranging the plasma density, temperature and current distribution.

1. Introduction

Due to the finiteness of electrical plasma resistivity the equilibrium state of tokamak plasmas is generally a state with plasma flow. Heating the plasma by means of neutral injection will increase the plasma flow velocities and let the flow effects become still more important. Correspondingly there arose an increased interest in plasma equilibria with flow mainly during the last years, [1−5].

The question arises how the resistively induced plasma flow across magnetic surfaces is affected by taking into account either nonlinear flow terms or ion viscosity or both.

2. A Global Invariance

Assuming quasineutrality and taking into account the nonlinear inertial terms and viscous forces, for steady states we have the momentum-equations

$$n m_e \, v_e \cdot \nabla v_e = - \nabla p_e - e n \, (E + v_e \times B)$$
$$- n^2 e^2 \, \mathbf{\eta} : (v_e - v_i) + F_e + R_T, \quad (2.1)$$

$$n m_i \, v_i \cdot \nabla v_i = \quad - \nabla p_i + e n \, (E + v_i \times B)$$
$$+ n^2 e^2 \, \mathbf{\eta} : (v_e - v_i) + F_i - R_T \quad (2.2)$$

for electrons and ions, respectively.

Here, $m_{i,e}$, $v_{i,e}$ and $p_{i,e}$ are the particle masses, flow velocities and pressures of ions and electrons, respectively. E and B are the electric and magnetic

fields. $\mathbf{\eta}$ is the electrical conductivity tensor, the $\mathbf{\eta}$-terms in $(2.1)-(2.2)$ representing the contribution of the viscous forces between electrons and ions due to a velocity difference $v_e - v_i \neq 0$. The thermal force R_T is a contribution of these due to a gradient of the collision frequency and is present even if $v_e = v_i$, [6]. In strong magnetic fields when the temperature gradient ∇T is perpendicular to the magnetic field, the thermal force becomes rather small, $R_T \sim B/B^2 \times \nabla T$ [7], and is frequently neglected. F_i and F_e are the viscous forces due to ion/ion and electron/electron collisions. In the presence of a magnetic field they are generally anisotropic forces.

Assuming $v_e/v_i \approx p_e/p_i = O(1)$ and neglecting terms of $O(m_e/m_i)$ against 1 (correspondingly F_e against F_i), from $(2.3)-(2.4)$ one obtains in the usual way

$$\nabla p - j \times B = F_i - n m_i \, v \cdot \nabla v, \quad (2.3)$$

$$\frac{1}{ne} R_T + \mathbf{\eta} : j = E + v \times B + \frac{\nabla p_e}{ne} - \frac{1}{ne} j \times B. \quad (2.4)$$

If F_i and/or the $v \cdot \nabla v$ term are not taken into account, instead of (2.4) one gets

$$\frac{1}{ne} R_T + \mathbf{\eta} : j = E + v \times B - \frac{\nabla p_i}{ne}. \quad (2.5)$$

$\nabla \cdot B = 0$ and axial symmetry imply generally

$$B = \Lambda \nabla \phi + \nabla \phi \times \nabla \psi, \quad (2.6)$$

where R, ϕ, z are cylindrical coordinates, $\Lambda = \Lambda(R, z)$, and $\psi = \psi(R, z)$ is the flux function of the poloidal

magnetic field. With

$$(v \times B) \cdot \nabla\phi = v \cdot \nabla\psi/R^2 = v_\psi B_{\mathrm{pol}}/R , \qquad (2.7)$$

$$(j \times B) \cdot \nabla\phi = (\nabla \times B) \cdot \frac{\nabla\psi}{R^2} = \frac{(\nabla\Lambda \times \nabla\phi) \cdot \nabla\psi}{R^2} \qquad (2.8)$$

the toroidal component of (2.5) yields

$$nv_\psi = \frac{n}{B_{\mathrm{pol}}} \left[\left(\frac{1}{ne} R_{\mathrm{T}} + \boldsymbol{\eta} : j \right)_\phi - E_\phi \right]$$
$$+ \frac{1}{e} (\nabla\Lambda \times \nabla\phi) \cdot \frac{\nabla\psi}{|\nabla\psi|} . \qquad (2.9)$$

The last term on the right hand side, the Hall term, is either due to F_i or $nm_\mathrm{i} v \cdot \nabla v$ or to both since, starting with (2.5), one obtains only the first term. Therefore, the electromagnetic field configuration, the temperature and density profile (i.e. the first term) being given, there is a direct influence of F_i and/or $nm_\mathrm{i} v \cdot \nabla v$ on nv_ψ only if $(\nabla\Lambda \times \nabla\phi) \cdot \nabla\psi \not\equiv 0$.

Although the Hall-term is usually considered to be small, it appears doubtful that it can be neglected in (2.9) since the other terms, which may, separately considered, be much larger, add up to a quantity which becomes even zero for $\boldsymbol{\eta} \to 0$.

The total flow F through a magnetic surface $\psi = $ const is given by the surface integral

$$F = \int\limits_{\psi = \mathrm{const}} nv_\psi \, dS \qquad (2.10)$$

$$= \int\limits_{\psi = \mathrm{const}} \frac{n}{B_{\mathrm{pol}}} \left[\left(\frac{1}{ne} R_{\mathrm{T}} + \boldsymbol{\eta} : j \right)_\phi - E_\phi \right] dS .$$

The $(\nabla\Lambda \times \nabla\phi) \cdot \nabla\psi$-term in (2.11) does not yield a contribution of F since the total current out of a magnetic surface is zero,

$$\int\limits_{\psi = \mathrm{const}} (\nabla\Lambda \times \nabla\phi) \cdot \frac{\nabla\psi}{|\nabla\psi|} \, dS$$
$$= \int\limits_{\psi = \mathrm{const}} [\nabla \times (\phi \nabla\Lambda)] \cdot dS = 2\pi \oint\limits_{\psi = \mathrm{const}} \nabla\Lambda \cdot dl_{\mathrm{pol}} = 0 .$$

We have the result that relation (2.10) for the plasma flow remains invariant when the nonlinear flow terms and/or ion viscosity are "gradually switched on". In other words: there is no *direct* contribution of these terms to F. However, this does not mean, that diffusion out of tokamaks would remain unaffected by them. They may still have an *indirect* effect in that they lead to changes in the

density and current distribution via the other components of Ohm's law and the momentum equation, which have not been considered here.

3. Conditions for Local Invariance

The flow terms under consideration enter the local equation (2.9) for nv_ψ if and only if $(\nabla\phi \times \nabla\psi) \cdot \nabla\Lambda \not\equiv 0$. If $(\nabla\phi \times \nabla\psi) \cdot \nabla\Lambda \equiv 0$, like in static equilibrium ($v \equiv 0$) we have

$$\Lambda = \Lambda(\psi) . \qquad (3.1)$$

3.1 Case of Negligible Ion-Viscosity

We shall now look for the consequences of (3.1) assuming first that ion viscosity can be neglected. Using (2.8), the ϕ-component of (2.3) yields in cylindrical coordinates R, ϕ, z

$$e_\phi \cdot (v \cdot \nabla) v = v \cdot \nabla v_\phi + v_R v_\phi/R = 0 \qquad (3.2)$$

e_ϕ being the unit vector in ϕ-direction. Due to $v \cdot \nabla v = \nabla v^2/2 - v \times (\nabla \times v)$ this equation is automatically satisfied if $\nabla \times v \equiv 0$, or if $v_\phi = 0$. The most general solution of (3.2) is: v_{pol} arbitrary and $v_\phi = g/R$, where g must be constant along the streamlines of the poloidal flow field v_{pol}. For all flows of this kind, (2.9) constitutes a local invariance with the consequence that even the local flow is influenced only *indirectly* by nonlinear flow terms and ion viscosity.

3.2 Scalar Ion Viscosity

In the presence of magnetic fields there exists an anisotropic relationship between the viscous force F_i and the ion velocity v_i, [6, 3]. This relation is rather complicated, and we shall therefore restrict our consideration to the simplified case of a scalar hydrodynamic viscosity. According to [8] we set

$$F_\mathrm{i} = (\lambda + \tfrac{4}{3}\mu) \nabla\nabla \cdot v - \mu \nabla \times (\nabla \times v) . \qquad (3.3)$$

With this and (2.8), the ϕ-component of (2.3) yields

$$\varrho_\phi \cdot (v \cdot \nabla) v = \frac{\mu}{nm_\mathrm{i}} \varrho_\phi \cdot [\nabla \times (\nabla \times v)] \qquad (3.4)$$

$$= v \cdot \nabla v_\phi + \frac{v_R v_\phi}{R} + \frac{\mu}{nm_\mathrm{i}} \left(\frac{v_\phi}{R^2} - \Delta v_\phi \right) = 0 .$$

Again $\nabla \times \boldsymbol{v} \equiv 0$ or $v_\phi \equiv 0$ are sufficient conditions for (3.4) to be satisfied. More generally, in accordance with Sect. 3.1 we set $v_\phi = \tilde{g}/R$ and obtain

$$\Delta \tilde{g} - \frac{2}{R}\frac{\partial}{\partial R}\tilde{g} - \frac{n m_\mathrm{i}}{\mu}\,\boldsymbol{v}\cdot\nabla\tilde{g} = 0 \qquad (3.5)$$

as an equation for determining $\tilde{g}$, v_pol being again arbitrary. Only if the toroidal flow velocity violates condition (3.5) there is a *direct* influence of scalar ion viscosity on the local diffusion.

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